COEFFICIENT INEQUALITIES RELATED WITH A SUBCLASS OF UNIVALENT FUNCTIONS DEFINED BY RUSCHEWEYH OPERATOR

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ABSTRACT. *Making use of a Ruscheweyh operator, we consider a new subclass of univalent functions which generalize the classes of strongly starlike and strongly convex functions. For this class we shall investigate the sharp upper bound of the second Hankel determinant and the well-known Fekete-Szegö inequality.*

Keywords: Strongly starlike functions, strongly convex functions, Ruscheweyh derivative, Second Hankel determinant.

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1. INTRODUCTION

Let A denote the class of functions $f(z)$ which are analytic in the region $\mathbb{A}=\{z\in \mathbb{C}:\ |z|<1\}$: and normalized by the conditions $f(0)=0$ and $f'(0)-1=0.$ Therefore, for $f(z)\in A,$ one has

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n,\ z\in\mathbb{A}, \tag{1.1}$$

while S denotes the class of normalized univalent functions. Also, let $\tilde{S}(\xi)$ and $\tilde{C}(\xi)$ are the classes of strongly starlike and strongly convex functions of order ξ $(0<\xi\le 1)$ respectively, that is,

$$\tilde{S}(\xi)=\left\{f(z)\in S:\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|\le\frac{\xi\pi}{2},\ z\in\mathbb{A}\right\}, \tag{1.2}$$

and

$$\tilde{C}(\xi)=\left\{f(z)\in S:\ \left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|\le\frac{\xi\pi}{2},\ z\in\mathbb{A}\right\}. \tag{1.3}$$

We note that for $\xi=1,$ the two classes defined in (1.2) and (1.3) reduces to the usual well-known classes of starlike S^* and convex C respectively.

For $f(z)\in A$ given by (1.1) and $g(z)\in A$ given by

$$g(z)=z+\sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_n\ z^n\ \ (z\in\mathbb{A}).$$

Using the concepts of Hadamard product, Ruscheweyh [18] in 1975 introduced a differential operator D^{δ} $(\delta>-1)$ which is defined by

$$D^{\delta}f(z)=\frac{z}{(1-z)^{\delta+1}}*f(z)$$
$$=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+\delta)}{(n-1)!\ \Gamma(\delta+1)}a_n\ z^n, \tag{1.4}$$

where $\Gamma(n+\delta)$ denote the Euler's functions with

$$\Gamma(n+\delta)=\delta(\delta+1)\cdots(\delta+n-1)\Gamma(\delta). \tag{1.5}$$

With the help of this operator several authors introduced and studied many subclasses of analytic and univalent functions, see for detail [4,8,19,11,12,17,19]. We now define a subclass $W_b(\delta,\xi)$ of univalent functions using this operator as follows:

$$W_b(\delta,\xi)=\left\{f(z)\in S\ :\ \left|\arg\left\{1+\frac{2}{b}\left(\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}-1\right)\right\}\right|\le\frac{\xi\pi}{2}\right\}.$$

We note that $W_1(1,\xi)\equiv\tilde{C}(\xi)$ and $W_2(0,\xi)\equiv\tilde{S}(\xi)$. Further for $\xi=1$, these classes reduces to the usual classes of starlike and convex functions respectively.

In 1933, Fekete and Szegö [5] proved a remarkable result known nowadays as Fekete-Szegö inequality which states that for $f(z)\in S$ given by (1.1),

$$|a_3-\mu a_2^2|\le\begin{cases}3-4\mu, & \text{if } \mu\le 0\\ 1+2e^{-2\mu(1-\mu)}, & \text{if } 0\le\mu\le 1\\ -3+4\mu, & \text{if } \mu\ge 1.\end{cases}$$

This inequality is sharp. After this result various authors studied the sharp upper bounds of the functional $|a_3-\mu a_2^2|$ for different subclasses of the class A of analytic functions and the class S of univalent, however we mention only few names here such as Keogh and Merkes [9], Frasin and Darus [6], Cho and Owa [12], Ali, Ravichandran and Seenivasagan [2], Raza, Arif and Darus [11], Ravichandran, Gangadharan and Darus [16] and Srivastava, Mishra and Das [20]. The purpose of the present article is to obtain sharp Fekete-Szegö inequalities and upper bound of second Hankel determinant for functions belonging to the class $W_b(\delta,\xi)$.

For our main results we need the following lemmas.

Lemma 1.1. If $q(z)$ is analytic in $\mathbb{A}$ and satisfy $\operatorname{Re} q(z) > 0$ for $z \in \mathbb{A}$, with

$$q(z) = 1 + \sum_{n=2}^{\infty} p_n z^n, \quad (z \in \mathbb{A}).$$

Then

(*i*). $|p_n| \le 2$ for $n \ge 1$.

(*ii*). $\left|p_2 - \frac{1}{2} p_1^2\right| \le 2 - \frac{1}{2}|p_1|^2$.

The first part was proved by Caratheodory [3] and the second part can be found in [15].

Lemma 1.2 [7]. If $p(z) \in P$ and of the form (1.4), then

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right)$$

for some $x, |x| \le 1$, and

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right) p_1 x - \left(4 - p_1^2\right) p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z$$

for some $z, |z| \le 1$.

Lemma 1.3. If $f(z) \in W_b(\delta, \xi)$ and given by (1.1), then

$$|a_2| \le |b|\xi.$$

$$|a_3| \le \begin{cases} \dfrac{\left[(\delta+1)|b|+1\right]|b|\xi^2}{\delta+2}, & \text{if } \xi \ge \dfrac{1}{(\delta+1)|b|+1} \\ \dfrac{|b|\xi}{\delta+2}, & \text{if } \xi \le \dfrac{1}{(\delta+1)|b|+1}, \end{cases}$$

where $\delta > -1, \xi \in (0,1]$ and $b \in \mathbb{C}$..

Proof. Let us suppose

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1} f(z)}{D^{\delta} f(z)} = \left[q(z)\right]^{\xi},$$

where $q(z)$ is analytic in $\mathbb{A}$ with $q(0) = 1$.

By using the following identity

$$(\delta+1) D^{\delta+1} f(z) = \delta D^{\delta} f(z) + z\left(D^{\delta} f(z)\right)',$$

we can write (1.7) in a simplified form as

$$\delta D^{\delta} f(z) + z\left(D^{\delta} f(z)\right)' = (\delta+1)\left[\frac{b}{2}\left\{[q(z)]^{\xi} - 1\right\}\right] D^{\delta} f(z).$$

For, the Taylor series expansion of $D^{\delta} f(z)$ being

$$D^{\delta} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\delta)}{(n-1)!\,\Gamma(\delta+1)} a_n z^n, \quad z \in \mathbb{A},$$

and also

$$\left[q(z)\right]^{\xi} = \left[1 + \sum_{n=2}^{\infty} p_n z^n\right]^{\xi}.$$

So from (1.8), (1.9) and (1.10), we get

$$z + \left(\delta a_2 + 2a_2\right) z^2 + \left[\frac{\delta(\delta+2)}{2} a_3 + \frac{3}{2}(\delta+2) a_3\right] z^3 + \cdots =$$

$$z + \left[(\delta+1) a_2 + \frac{b}{2}\xi p_1\right] z^2 +$$

$$\left[\frac{(\delta+1)(\delta+2)}{2} a_3 + \frac{b}{2}\xi(\delta+1) a_2 + \frac{b}{4}\xi(\xi-1) p_1^2 + \frac{b}{2}\xi p_2\right] z^3 + \cdots$$

Comparing the coefficients of like terms, we have

$$a_2 = \frac{b}{2}\xi p_1,$$

which implies from Lemma 1.1 (i) that

$$|a_2| \le |b|\xi,$$

and the coefficient of z^3 is

$$a_3 = \frac{1}{(\delta+2)}\left[\frac{bp_1^2}{4}\left\{b(\delta+1)+1\right\}\xi^2 + \frac{b}{2}\left(p_2 - \frac{p_1^2}{2}\right)\xi\right].$$

This further implies by using Lemma 1.1 (ii), we have

$$|a_3| \le \frac{1}{(\delta+2)}\left\|\left[\frac{|b||p_1|^2}{4}\left\{b(\delta+1)+1\right\}\xi^2 + \frac{|b|}{2}\left(2 - \frac{|p_1|^2}{2}\right)\xi\right]\right\|.$$

After simplification, we get

$$|a_3| \le \begin{cases} \dfrac{\left[(\delta+1)|b|+1\right]|b|\xi^2}{\delta+2}, & \text{if } \xi \ge \dfrac{1}{(\delta+1)|b|+1} \\ \dfrac{|b|\xi}{\delta+2}, & \text{if } \xi \le \dfrac{1}{(\delta+1)|b|+1} \end{cases}.$$

2. Main Results

Theorem 2.1. Let $f(z) \in W_b(\delta, \xi)$ and $\xi \in (0,1]$. Then for the complex μ

$$\left|a_3 - \mu a_2^2\right| \le \frac{|b|\xi}{\delta+2} \max\left[1, \left|\xi\left\{(\delta+1)b + 1 - \mu(\delta+2)\right\}\right|\right]$$

For each μ there is a function in $W_b(\delta, \xi)$ such that equality holds.

Proof. From (1.11) and (1.12), we may write

$$a_3 - \mu a_2^2 = \frac{\left[(\delta+1)b^2 + b - \mu b^2(\delta+2)\right]\xi^2 p_1^2}{4(\delta+2)} + \frac{b\xi}{2(\delta+2)}\left(p_2 - \frac{p_1^2}{2}\right). \tag{2.1}$$

Using Lemma 1.1 (ii), we have

$$\left|a_3-\mu a_2^2\right| \le \frac{\left|\left[\{(\delta+1)-\mu(\delta+2)\}b^2+b\right]\xi^2 p_1^2\right|}{4(\delta+2)}+\frac{|b|\xi}{2(\delta+2)}\left(2-\frac{p_1^2}{2}\right)$$

$$=\frac{|b|\xi}{\delta+2}+\frac{\left[\left\{(\delta+1)|b|^2+|b|-|\mu||b|^2(\delta+2)\right\}\xi^2-|b|\xi\right]|p_1|^2}{4(\delta+2)}. \tag{2.2}$$

Let us take

$$k(\delta)=\left[\frac{(\delta+1)|b|^2+|b|-|\mu||b|^2(\delta+2)}{2}\right]\xi^2$$

Then (2.2) implies

$$\left|a_3-\mu a_2^2\right| \le \frac{|b|\xi}{\delta+2}+\frac{\left[k(\delta)-\frac{|b|\xi}{2}\right]|p_1|^2}{2(\delta+2)}. \tag{2.3}$$

If we take

$$k(\delta)-\frac{|b|\xi}{2}\le 0,$$

then (2.3) becomes

$$\left|a_3-\mu a_2^2\right| \le \frac{|b|\xi}{\delta+2}.$$

Now when

$$k(\delta)-\frac{|b|\xi}{2}\ge 0,$$

then, by using Lemma 1.1 (i), (2.3) can be written as

$$\left|a_3-\mu a_2^2\right| \le \left[\frac{(\delta+1)|b|^2+|b|-|\mu||b|^2(\delta+2)}{(\delta+2)}\right]\xi^2.$$

Hence

$$\left|a_3-\mu a_2^2\right| \le \begin{cases}\left[(\delta+1)|b|^2+|b|-|\mu||b|^2(\delta+2)\right]\dfrac{\xi^2}{\delta+2}, & \text{for } k(\delta)\ge\dfrac{|b|\xi}{2}\\[2mm] \dfrac{|b|\xi}{(\delta+2)}, & \text{for } k(\delta)\le\dfrac{|b|\xi}{2}.\end{cases}$$

Equality occurs for the functions in $W_b(\delta,\xi)$ given by

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}=1+\frac{b}{2}\left[\left(\frac{1+z^2}{1-z^2}\right)^{\xi}-1\right],$$

or

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}=1+\frac{b}{2}\left[\left(\frac{1+z}{1-z}\right)^{\xi}-1\right].$$

Taking $b=2$ and $\delta=0$ in Theorem 2.1, we get the following result.

Corollary 2.2. Late $f(z)\in S^*(\xi)$. Then for the complex μ

$$\left|a_3-\mu a_2^2\right| \le \xi \max\left[1,|3-4\mu|\xi\right].$$

This result is sharp.

Making $b=1=\delta$ in Theorem 2.1, we get corollary 2.3.

Corollary 2.3. Let $f(z)\in C(\xi)$. Then for complex μ

$$\left|a_3-\mu a_2^2\right| \le \frac{\xi}{3}\max\left[1,|3-\mu|\xi\right].$$

This result is sharp.

In the next result, we suppose that μ is real.

Theorem 2.4. Let $f(z)\in W_b(\delta,\xi)$ and $\xi\in(0,1]$. Then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases}\left[\dfrac{(\delta+1)|b|^2+|b|-\mu|b|^2(\delta+2)}{\delta+2}\right]\xi^2, & \text{for } \mu\le\dfrac{(\delta+1)|b|\xi+\xi-1}{(\delta+2)|b|\xi},\\[2mm] \dfrac{|b|\xi}{(\delta+2)}, & \text{for } \dfrac{(\delta+1)|b|\xi+\xi-1}{(\delta+2)|b|\xi}\le\mu\le\dfrac{(\delta+1)|b|\xi+\xi+1}{(\delta+2)|b|\xi},\\[2mm] \dfrac{\mu|b|^2(\delta+2)-(\delta+1)|b|^2-|b|}{\delta+2}\xi^2, & \text{for } \mu\ge\dfrac{(\delta+1)|b|\xi+\xi+1}{(\delta+2)|b|\xi}.\end{cases}$$

For each μ there is a function in $W_b(\delta,\xi)$ such that equality holds.

Proof. By virtue of (2.3), we have to consider the following two cases.

Case (i). Assume that

$$(\delta+2)|b|\mu\le(\delta+1)|b|+1.$$

Then in this case Lemma 1.1 (ii) and (2.1) give

$$\left|a_3-\mu a_2^2\right| \le \frac{\left|\left[\{(\delta+1)-\mu(\delta+2)\}|b|^2+|b|\right]\xi^2|p_1|^2\right|}{4(\delta+2)}+\frac{|b|\xi}{2(\delta+2)}\left(2-\frac{|p_1|^2}{2}\right)$$

$$=\frac{|b|\xi}{\delta+2}+\frac{\left[\left\{(\delta+1)|b|^2+|b|-|\mu||b|^2(\delta+2)\right\}\xi^2-|b|\xi\right]|p_1|^2}{4(\delta+2)}$$

and so, using Lemma 1.1 (i), we obtain

$$\left|a_3-\mu a_2^2\right| \le \begin{cases}\dfrac{(\delta+1)|b|^2+|b|-\mu|b|^2(\delta+2)}{\delta+2}\xi^2, & \text{if } \mu\le\dfrac{(\delta+1)|b|\xi+\xi-1}{(\delta+2)|b|\xi},\\[2mm] \dfrac{|b|\xi}{(\delta+2)}, & \text{if } \dfrac{(\delta+1)|b|\xi+\xi-1}{(\delta+2)|b|\xi}\le\mu\le\dfrac{(\delta+1)|b|\xi+\xi+1}{(\delta+2)|b|\xi},\end{cases} \tag{2.4}$$

Equality occurs if $p_1=p_2=2$ and $p_1=0, p_2=2$, respectively in (2.1).

Case (ii) Now if

$$(\delta+2)|b|\mu\ge(\delta+1)|b|+1,$$

then, from Lemma 1.1 (ii) and (2.1), we have

$$\left|a_3-\mu a_2^2\right| \le \frac{\left|\left[\{\mu(\delta+2)-(\delta+1)\}|b|^2-|b|\right]\xi^2|p_1|^2\right|}{4(\delta+2)}+\frac{|b|\xi}{2(\delta+2)}\left(2-\frac{|p_1|^2}{2}\right)$$

$$=\frac{|b|\xi}{\delta+2}+\frac{\left[\left\{|\mu||b|^2(\delta+2)-(\delta+1)|b|^2-|b|\right\}\xi^2-|b|\xi\right]|p_1|^2}{4(\delta+2)},$$

and so by using Lemma 1.1 (i), we get

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{\mu |b|^2(\delta+2)-(\delta+1)|b|^2-|b|}{\delta+2}\xi^2, \text{ if } \mu \geq \dfrac{(\delta+1)|b|\xi+\xi-1}{(\delta+2)|b|\xi}, \\ \dfrac{|b|\xi}{(\delta+2)}, \quad \text{if } \dfrac{(\delta+1)|b|+1}{(\delta+2)|b|} \leq \mu \leq \dfrac{(\delta+1)|b|\xi+\xi+1}{(\delta+2)|b|\xi}. \end{cases} \tag{2.5}$$

Equality is attained when we choose $p_1=0,\ p_2=2$ and $p_1=2i,\ p_2=-2$ respectively in (2.1). Thus, from (2.4) and (2.5), we get the required result.

Taking $b=2, \delta=0$ in Theorem 2.4, we get the following result.

Corollary 2.5. Late $f(z) \in S^*(\xi)$. Then for the complex μ

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} (3-4\mu)\xi^2, \text{ if } \mu \leq \dfrac{3\xi-1}{4\xi}, \\ \xi, \quad \text{if } \dfrac{3\xi-1}{4\xi} \leq \mu \leq \dfrac{3\xi+1}{4\xi}, \\ (4\mu-3)\xi^2, \text{ if } \mu \geq \dfrac{3\xi+1}{4\xi}. \end{cases}$$

By putting $b=1=\delta$ in Theorem 2.4, we obtain the following result.

Corollary 2.6. Let $f(z) \in C(\xi)$. Then for real μ

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} (1-\mu)\xi^2, \text{ if } \mu \leq \dfrac{3\xi-1}{3\xi}, \\ \dfrac{\xi}{3} \text{ if } \dfrac{3\xi-1}{3\xi} \leq \mu \leq \dfrac{3\xi+1}{3\xi}, \\ (\mu-1)\xi^2, \text{ if } \mu \geq \dfrac{3\xi+1}{\xi}. \end{cases}$$

Theorem 2.7. Let $f(z) \in W_b(\delta,\xi)$. Then

$$|a_3|-|a_2| \leq \frac{|b|\xi}{(\delta+2)}, \quad \text{if } \xi \leq \frac{3}{\delta|b|-|b|+3}.$$

Proof. We easily write

$$|a_3|-|a_2| \leq \left|a_3-\frac{2}{3}a_2^2\right|+\frac{2}{3}|a_2|^2-|a_2|$$

Since for

$$\frac{(\delta+1)|b|\xi+\xi-1}{(\delta+2)|b|\xi} \leq \frac{2}{3},$$

that is,

$$\xi \leq \frac{3}{\delta|b|-|b|+3}.$$

Therefore, by using Theorem 2.2, we have

$$|a_3|-|a_2| \leq \frac{|b|\xi}{(\delta+2)}+\frac{2}{3}|a_2|^2-|a_2| = \lambda(x),$$

Where $x=|a_2| \in \left[0,|b|\xi\right]$. But since $\lambda(x)$ attains its maximum value at $x=0$. Therefore, we get the required result.

Theorem 2.8. Let $f(z) \in W_b(\delta,\xi)$. Then

$$\left|a_2a_4-a_3^2\right| \leq \frac{|b|^2\xi^2}{(\delta+2)^2}.$$

Proof. By using the values of a_2, a_3 and a_4 we have

$$a_2a_4-a_3^2 = \frac{-1}{(\delta+2)^2}\left[\left\{\frac{(\delta+1)b^2+b}{4}\right\}p_1^2\xi^2+\frac{b}{2}\left(p_2-\frac{p_1^2}{2}\right)\xi\right]^2+\left(\frac{b}{2}\xi p_1\right)$$

$$\left[\left\{\frac{\xi b}{6}-\left(b-\frac{3b^2(\delta+1)}{4}\right)\xi^2+\left(\frac{b}{3}-\frac{3b^2(\delta+1)}{4}+\frac{b^3(\delta+1)^2}{4}\right)\xi^3\right\}\frac{2p_1^3}{(\delta+2)(\delta+3)}\right.$$

$$\left.+\left\{-b\xi+b\xi^2+\frac{3\xi^2b^2(\delta+1)}{4}\right\}p_1p_2+\frac{1}{2}b\xi p_3\right],$$

Or, equivalently

$$\left|a_2a_4-a_3^2\right| = \left|Ap_1^4+Bp_1^2p_2+Cp_2^2+Dp_1p_3\right|$$

with

$$A = \frac{-1}{(\delta+2)^2}\left(\frac{(\delta+1)^2b^2\xi^4}{16}+\frac{b^2\xi^4}{16}+\frac{(\delta+1)b^3\xi^4}{8}+\frac{b^2\xi^2}{16}-\frac{(\delta+1)b^3\xi^3}{4}-\frac{b^2\xi^3}{8}\right)+$$

$$\frac{1}{(\delta+2)(\delta+3)}\left(\frac{b^2\xi^2}{6}-\frac{b^2\xi^3}{4}+\frac{3(\delta+1)b^3\xi^3}{16}+\frac{b^2\xi^4}{12}+\frac{3(\delta+1)b^3\xi^4}{16}+\frac{(\delta+1)^2b^4\xi^4}{16}\right),$$

$$B = \frac{1}{(\delta+2)^2}\left(\frac{b^2\xi^2}{4}-\frac{(\delta+1)b^3\xi^3}{4}-\frac{b^2\xi^3}{4}\right)-\frac{b^2\xi^2}{2}+\frac{b^2\xi^3}{2}+\frac{3(\delta+1)b^3\xi^3}{8},$$

$$C = -\frac{b^2\xi^2}{4(\delta+2)^2}, \quad D = \frac{b^2\xi^2}{2}.$$

Putting the values of p_2 and p_3 from Lemma 1.2 and suppose that $p>0$ with $p_1=p\in[0,2]$, we obtain

$$\left|a_2a_4-a_3^2\right| = \left|(4A+2B+C+D)\frac{p^4}{4}+\frac{1}{2}Dp\left(4-p^2\right)\left(1-|x|^2\right)z+\right.$$

$$\left.\frac{1}{2}(B+C+D)p^2\left(4-p^2\right)x-\frac{1}{4}\left\{Dp^2-C\left(4-p^2\right)\right\}\left(4-p^2\right)x^2\right|$$

Using triangle inequality and taking $|x|=\rho$ and $|z|<1$, we have

$$\left|a_2a_4-a_3^2\right| \leq |4A+2B+C+D|\frac{p^4}{4}+\frac{1}{2}|B+C+D|p^2\left(4-p^2\right)\rho+$$
$$\frac{1}{2}Dp\left(4-p^2\right)\left(1-\rho^2\right)++\frac{1}{4}\left\{Dp^2-C\left(4-p^2\right)\right\}\left(4-p^2\right)\rho^2$$
$$=F(p,\rho), \text{ (say).}$$

Differentiating with respect to ρ, we get

$$\frac{\partial F(p,\rho)}{\partial \rho}=\frac{1}{2}|B+C+D|\,p^2\left(4-p^2\right)-Dp\left(4-p^2\right)\rho+\frac{1}{2}\left\{Dp^2-C\left(4-p^2\right)\right\}\left(4-p^2\right)\rho.$$

It is clear that $\frac{\partial F(p,\rho)}{\partial \rho}>0,$ which shows that $F(p,\rho)$ is an increasing function on the interval $[0,1].$ This implies that maximum value occurs at $\rho=1.$ Therefore $F(p,\rho)=F(p,1)=N(p)$ (say).

$$N(p)=|4A+2B+C+D|\frac{p^4}{4}+\frac{1}{2}|B+C+D|\,p^2\left(4-p^2\right)++\frac{1}{4}\left\{Dp^2-C\left(4-p^2\right)\right\}\left(4-p^2\right)$$

Differentiate again with respect to p, we have

$$N'(p)=\left[|4A+2B+C+D|-2|B+C+D|-D-C\right]p^3+\left[4|B+C+D|+2D+4C\right]p$$

Again, differentiate with respect to p, we get

$$N''(p)=3\left[|4A+2B+C+D|-2|B+C+D|-D-C\right]p^2+\left[4|B+C+D|+2D+4C\right]$$

Now $N''(p)<0$ for $p=0,$ so the maximum value occurs at $p=0$ and hence

$$\left|a_2a_4-a_3^2\right|\leq\frac{|b|^2\xi^2}{(\delta+2)^2}.$$

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